Wringing of cubic polynomials with critical orbit relations

Shizuo Nakane *

In this note, wring deformation of cubic polynomials is investigated. Especially, turning deformation of maps with critical orbit relations is studied. It is described via the fractional Dehn twist and the hemidromy is calculated.

1 Introduction

Let $\mathcal{P}_d$ be the family of monic centered polynomials of degree $d \geq 2$. For $P \in \mathcal{P}_d$, let φ_P be its Böttcher coordinate and $g_P(z) = \log_+ |\varphi_P(z)|$ be the Green function for P. For a complex number $u \in \mathbb{H}_+ = \{u = s + it \in \mathbb{C}, s > 0\}$, put $f_u(z) = z|z|^{u-1}$ and we define a P-invariant almost complex structure σ_u by

$$\sigma_u = \begin{cases} (f_u \circ \varphi_P)^* \sigma_0 \text{ on } U_P, \\ \sigma_0 \text{ on } K(P). \end{cases}$$

Then, by the Measurable Riemann Mapping Theorem, σ_u is integrated by a qc-map F_u and $P_u = F_u \circ P \circ F_u^{-1} \in \mathcal{P}_d$. Thus we define a holomorphic map $W_P : \mathbb{H}_+ \to \mathcal{P}_d$ by $W_P(u) = P_u$. This qc-deformation, what we call *wringing*, was originally defined and developed in Branner-Hubbard [BH1, BH2]. See also Branner [B].

The Böttcher coordinate φ_{P_u} of P_u is equal to $f_u \circ \varphi_P \circ F_u^{-1}$. Since P_u is hybrid equivalent to P, it holds $P_u \equiv P$ for $P \in \mathcal{C}_d$, the *connectedness locus*. The wringing yields special type of curves in the *escape locus* $\mathcal{E}_d$. For $P \in \mathcal{E}_d$, we define the *stretching ray* through P by

$$R(P) = W_P(\mathbb{R}_+) = \{P_s; s \in \mathbb{R}_+\},$$

and the *turning curve* through P by

$$T(P) = W_P(1 + i\mathbb{R}) = \{P_{1+it}; t \in \mathbb{R}\}.$$

In case $d = 2$, stretching rays and turning curves coincide respectively with the *external rays* and *equipotential curves* for the Mandelbrot set.

In this note, we investigate the *monodromy* or the *hemidromy* map. The monodromy map is the first return map along the turning curve. These maps turn out to be obtained from the pull-backs by φ_P of what we call the Dehn twist on an

*Professor, General Education and Research Center, Tokyo Polytechnic University
Received Sept. 13, 2006

annulus in the Böttcher coordinate. We will investigate the orbit by the hemidromy of some examples of cubic polynomials. It essentially depends on the *pattern* or the *critical portrait* of the map P. It turns out that the hemidromy map is not transitive among the maps with a given critical orbit relation. This is different from the case of the *central hyperbolic component* of cubic polynomials, i.e., those cubics with both critical points in the immediate basin of an attracting fixed point.

2 Turning of cubic polynomials

We consider the family of cubic polynomials :

$$P(z) = P_{a,b}(z) = z^3 - 3a^2 z + b; \quad a, b \in \mathbb{C}.$$

Note that the critical points are $\pm a$ and their *co-critical points* are $\mp 2a$ respectively. After [B], we prepare some notations. Level curves of the Green function g_P are called *equipotential curves*. If we put $G(P) = max\{g_P(a), g_P(-a)\}$ and $U_P = \{z \in \mathbb{C}; g_P(z) > G(P)\}$, then $\varphi_P : U_P \to \{|z| > \exp(G(P))\}$ is a conformal isomorphism. Thus the ray $R_P(t) = \varphi_P^{-1}(\{re^{2\pi it}; r > \exp(G(P))\})$ is called the *external ray* with *external angle* $t \in \mathbb{R}/\mathbb{Z}$. Since φ_P satisfies the functional equation $\varphi_P(P(z)) = (\varphi_P(z))^n$, we can extend it by this equation. Thus external rays can be also continued until they meet the precritical points. Just two external rays hit the critical point a. The map φ_P can be analytically continued to a neighborhood of the co-critical point $-2a$.

For $\rho > 0$, we define

$$\mathcal{S}_\rho = \{P \in \mathcal{E}_3; G(P) = \rho\}.$$

This set is invariant under turning.

Lemma 2.1. *$T(P) \subset \mathcal{S}_\rho$ if $P \in \mathcal{S}_\rho$.*

proof. Suppose $P \in \mathcal{S}_\rho$. We may assume $G(P) = g_P(a) = \rho$. Then $G(P_{1+it} = \log|\varphi_{P_{1+it}}(a_t)| = \log|f_{1+it}(\varphi_P(a))| = g_P(a) = \rho$. This completes the proof. □

In the sequel, we consider the subset of the parameter space :

$$\mathcal{H}_\rho = \{P \in \mathcal{S}_\rho; g_P(a) = G(P) > g_P(-a)\}.$$

Then $\varphi_P(-2a)$ is well defined there, hence we put $\Phi_\rho(P) = \dfrac{\varphi_P(-2a)}{|\varphi_P(-2a)|}$ and, for $\theta \in \mathbb{R}/\mathbb{Z}$, we define

$$F_\rho(\theta) = \{P \in \mathcal{H}_\rho; \Phi_\rho(P) = e^{2\pi i\theta}\}.$$

Then we have

Theorem 2.1. *([B], Theorem 6.2)*
The map $\Phi_\rho : \mathcal{H}_\rho \to S^1$ is a trivial topological fibration with fibers homeomorphic to the unit disk.

Thus each $F_\rho(\theta)$ is a disk transversal to the wringing leaves. Put $T_{\rho,\theta}(P,t) = W_P(1+i\dfrac{2\pi t}{\rho})$.

Lemma 2.2. *$T_{\rho,\theta} : F_\rho(\theta) \times \{t\} \cong F_\rho(\theta + t)$.*

proof. If $P \in F_\rho(\theta)$, it follows $\varphi_P(-2a) = \rho e^{2\pi i\theta}$. Put $P_t = W_P(1+i\dfrac{2\pi t}{\rho})$. Then

$$\begin{aligned}\varphi_{P_t}(-2a_t) &= f_{1+2\pi it/\rho}(\varphi_P(-2a)) \\ &= e^{\rho+2\pi i\theta}e^{\rho\cdot 2\pi it/\rho} \\ &= e^{\rho+2\pi i(\theta+t)},\end{aligned}$$

which completes the proof. □

Thus, if we put $t = 1$, we can define the *monodromy* $M_\rho : F_\rho(\theta) \to F_\rho(\theta)$ by $M_\rho(P) = W_P(1+i\dfrac{2\pi}{\rho})$. This map is the first return map to $F_\rho(\theta)$ along the turning curve through P.

3 Fractional Dehn twists

In this section, we describe the monodromy map M_ρ via the *fractional Dehn twist* on an annulus.

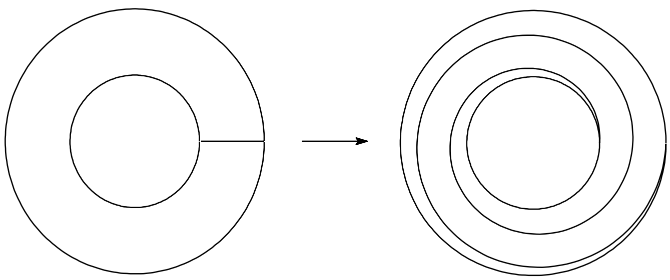
Let $A_r = \{z \in \mathbb{C}; r < |z| < r^3\}$ be an annulus. The fractional Dehn twist D_q of fraction $q \in \mathbb{Q}$ on A_r is just defined by

$$D_q(R\exp(2\pi it)) = R\exp(2\pi i(t + q\frac{\log R}{\log r})).$$

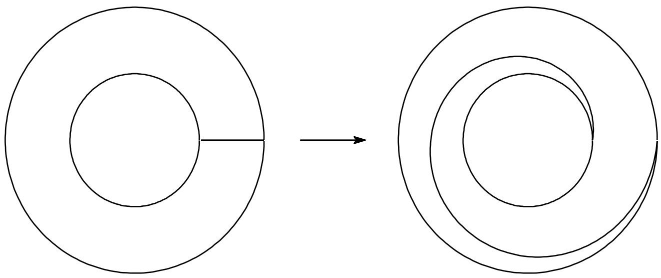
Put $u = 1 + \dfrac{2\pi i}{\log r}$. Then the map $f_u(z) = z|z|^{u-1}$ is a qc-map $A_r \to A_r$ of the form :

$$f_u(Re^{2\pi it}) = R\exp(2\pi i(t + \frac{\log R}{\log r})),$$

which is just the Dehn twist D_1 on the annulus A_r. See the following figure. It spirals twice around the origin.



Put $u' = 1 + \dfrac{\pi i}{\log r}$. Then the map $f_{u'}$ coincides with the Dehn twist $D_{1/2}$. The map $-f_{u'} = -D_{1/2}$ spirals just once as follows



and the map $z \mapsto f_u(z)$ is the second iterate of the map $z \mapsto -f_{u'}(z)$. Hence, if we put $H_\rho(P) = -W_P(1 + i\dfrac{\pi}{\rho})$, it follows $M_\rho = H_\rho^2$. The map $H_\rho : F_\rho(\theta) \to F_\rho(\theta)$ is called the *hemidromy*.

These Dehn twists on the annuli in the Böttcher coordinate are pull-backed by φ_P on the annuli in the dynamical plane. Consider the subset $\mathcal{A}_n = \{z \in \mathbb{C}; \dfrac{\rho}{3^{n+1}} < g_P(z) < \dfrac{\rho}{3^n}\}$. Each connected component of $\mathcal{A}_n$ is an annulus, which we call an annulus of level n.

Then P maps each annulus A of level n onto an annulus A' of level $n-1$ with some degree depending on whether the annulus A surrounds the critical point $-a$ or not. See Figure 1. The shadow region is an annulus A of level 0. P maps A homeomorphically onto A', an annulus of level -1.

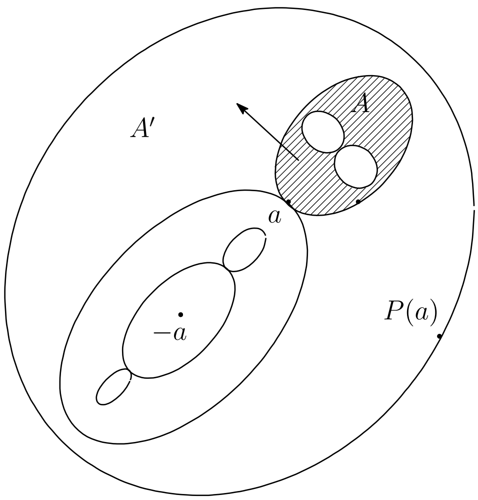


Figure 1: Annuli in the dynamical plane

Lemma 3.1. *([BH2])*
If $A \in \mathcal{A}_n$ surrounds $-a$, the map $P : A \to A'$ is a covering of degree two. The Dehn twist D_q on A' is pull-backed by P to the Dehn twist $D_{q/2}$ on A. Otherwise, the map $P : A \to A'$ is an isomorphism and D_q on A' is pull-backed to D_q on A.

4 Hemidromy of cubic polynomials

We will compute the hemidromy in some cases. Consider the space $F_\rho(2/3)$. For $P \in F_\rho(2/3)$, the external rays $R_P(0)$ and $R_P(1/3)$ hit the critical point a. Note that the wringing preserves the critical orbit relation. Hence, if $P \in F_\rho(2/3)$ satisfies $P^k(-a) = a$ for some $k > 0$, then so does $H_\rho(P)$. In case $k = 2$, there exist essentially three cubic polynomials. The following three figures describe the dynamical plane of those three polynomials. In those figures, equipotential curves and external rays through both critical points $\pm a$, a critical value $P(-a)$ and a co-critical point $2a$ with their angles are drawn.

Proposition 4.1. *The map P_1 of Figure 4 is fixed by the hemidromy. The maps P_2 of Figure 2 and P_3 of Figure 3 are interchanged by the hemidromy. Consequently, the monodromy fixes all these maps.*

proof. Consider the map P_2 of Figure 2. In its dynamical plane, there is an annulus A surrounded by the equipotential curves passing through a and $P(-a)$. This annulus surrounds $-a$. Hence by Lemma 3.1, the Dehn twist $-D_{1/2}$ on its image annulus in U_P is pull-backed to the Dehn twist $-D_{1/4}$ on A. By this Dehn twist, the critical value $P(-a)$ is mapped to the position just in Figure 3. Thus the hemidromy interchanges P_2 and P_3. (But the monodromy fixes them.)

On the other hand, in Figure 4, the corresponding annulus does not surrounds $-a$, hence the Dehn twist does not change the position of $P(-a)$ and P_1 is fixed by the hemidromy. This completes the proof. □

We can construct a map that is not fixed by the monodromy. Consider the map P_4 in Figure 5. This represents the annuli of level 2. The hemidromy changes the left one to the middle one. Thus the monodromy changes the left one to the right one. Thus $M_\rho(P_4) \neq P_4$ and $M_\rho^2(P_4) = P_4$. By the same way, we have

Theorem 4.1. *For any $k > 0$, there is a map that is M_ρ- periodic with period 2^k.*

References

[B] B. Branner: Turning around the connectedness locus. In: "*Topological methods in modern Mathematics.*" pp. 391–427. Houston, Publish or Perish, 1993

[BH1] B. Branner and J. Hubbard: The iteration of cubic polynomials. Part I: The global topology of parameter space. Acta Math. **160**, (1988), pp. 143–206.

[BH2] B. Branner and J. Hubbard: The iteration of cubic polynomials. Part II: Patterns and parapatterns. Acta Math. **169**, (1992), pp. 229–325.

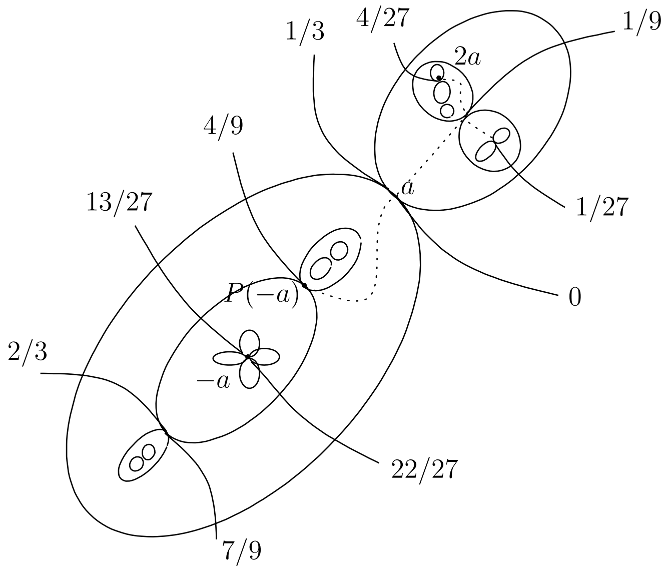


Figure 2: Map P_1 : mapped to P_2 by the hemidromy

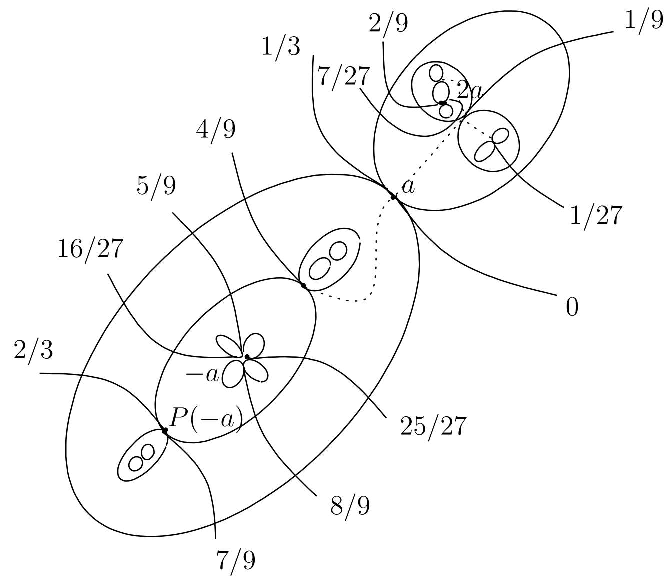


Figure 3: Map P_2 : mapped to P_1 by the hemidromy

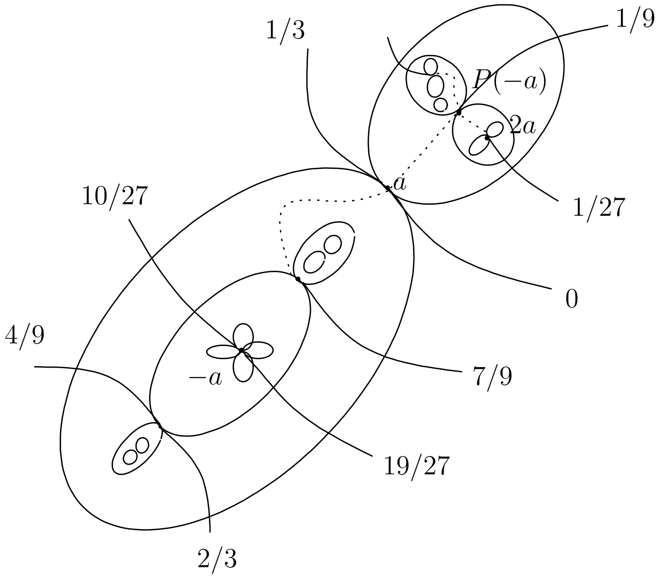


Figure 4: Map P_3 fixed by the hemidromy

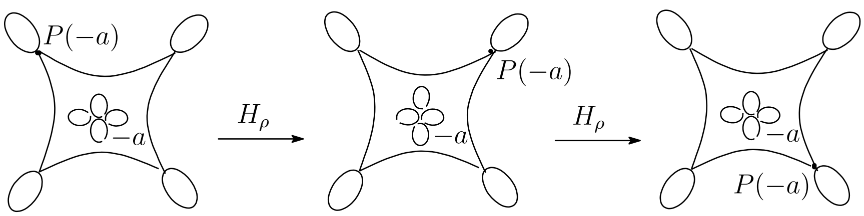


Figure 5: From left to right : P_4, $H_\rho(P_4)$, $M_\rho(P_4)$